

Fermionic Casimir densities in a conical space with a circular boundary and magnetic flux

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Abstract

The vacuum expectation value (VEV) of the energy-momentum tensor for a massive fermionic field is investigated in a (2+1)-dimensional conical spacetime in the presence of a circular boundary and an infinitely thin magnetic flux located at the cone apex. The MIT bag boundary condition is assumed on the circle. At the cone apex we consider a special case of boundary conditions for irregular modes, when the MIT bag boundary condition is imposed at a finite radius, which is then taken to zero. The presence of the magnetic flux leads to the Aharonov-Bohm-like effect on the VEV of the energy-momentum tensor. For both exterior and interior regions, the VEV is decomposed into boundary-free and boundary-induced parts. Both these parts are even periodic functions of the magnetic flux with the period equal to the flux quantum. The boundary-free part in the radial stress is equal to the energy density. Near the circle, the boundary-induced part in the VEV dominates and for a massless field the vacuum energy density is negative inside the circle and positive in the exterior region. Various special cases are considered.

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1 Introduction

Topological defects created after symmetry-breaking phase transitions play an important role in many fields of physics (for a review see [1]). They appear in different condensed matter systems including superfluids, superconductors, and liquid crystals. Moreover, the topological defects provide an important link between particle physics and cosmology. In particular, the cosmic strings are candidates to produce a number of interesting physical effects, such as the generation of gravitational waves, gamma ray bursts, and high-energy cosmic rays. In the simplest theoretical model, the geometry of a cosmic string outside its core is described by the planar angle deficit. Though this geometry is flat, the corresponding nontrivial topology leads

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to a number of interesting physical effects. In particular, the properties of the quantum vacuum are changed due to the modification of the zero-point fluctuations spectrum of quantum fields. Explicit calculations for the vacuum polarization by the cosmic string have been developed for different fields and in various spatial dimensions (see, for instance, [2]-[16]). The combined effects of the cosmic string topology and a coaxial cylindrical boundary on the polarization of the vacuum were studied in Refs. [17]-[20] for scalar, electromagnetic and fermionic fields.

In Refs. [21, 22] we have investigated the vacuum expectation value (VEV) of the fermionic current and the fermionic condensate induced by the vortex configuration of a gauge field in a (2+1)-dimensional conical space with a circular boundary. Continuing in this line of investigation, in the present paper we study the VEV of the energy-momentum tensor for a massive fermionic field with the MIT bag boundary condition. The imposition of the boundary condition induces shifts in the expectation values of physical characteristics of the vacuum state. This is the well known Casimir effect [23]. The expectation value of the energy-momentum tensor is among the most important characteristics of the vacuum. In addition to describing the physical structure of a quantum field at a given point, it acts as the source of gravity in the quasiclassical Einstein equations and plays an important role in modelling self-consistent dynamics involving the gravitational field. In considering the expectation value of the energy-momentum tensor we shall assume the presence of a magnetic flux. The interaction of a magnetic flux tube with a fermionic field gives rise to a number of interesting phenomena, such as the Aharonov-Bohm effect, parity anomalies, formation of a condensate, and generation of exotic quantum numbers. For background Minkowski spacetime, the combined effects of the magnetic flux and boundaries on the vacuum energy have been studied in Refs. [24, 25].

Field theories in 2+1 dimensions provide simple models in particle physics. Related theories also arise in the long-wavelength description of certain planar condensed matter systems, including models of high-temperature superconductivity. They exhibit a number of interesting effects, such as parity violation, flavor symmetry breaking, and fractionalization of quantum numbers (see Refs. [26]-[33]). An important aspect is the possibility of giving a topological mass to the gauge bosons without breaking gauge invariance. An interesting application of Dirac theory in 2+1 dimensions recently appeared in nanophysics. The long-wavelength description of the electronic states in a graphene sheet can be formulated in terms of the Dirac-like theory of massless spinors in (2+1)-dimensional spacetime with the Fermi velocity playing the role of the speed of light (for a review see Ref. [34]). One-loop quantum effects induced by the nontrivial topology of graphene made cylindrical and toroidal nanotubes have been recently considered in Ref. [35]. The vacuum polarization in graphene with a topological defect is investigated in Ref. [36] within the framework of a long-wavelength continuum model.

The outline of the paper is as follows. In the next section we consider the geometry of a boundary-free conical space with an infinitesimally thin magnetic flux placed at the apex of the cone. At the cone apex, a special case of boundary conditions is considered, when the MIT bag boundary condition is imposed at a finite radius, which is then taken to zero. The renormalized VEV of the energy-momentum tensor is evaluated. Integral representations are provided for the energy density and vacuum stresses. In Sect. 3, we consider the VEV of the energy-momentum tensor in the region inside a circular boundary with the MIT bag boundary condition. The VEV is decomposed into boundary-free and boundary induced parts. A rapidly convergent integral representation for the latter is obtained. A similar investigation for the region outside a circular boundary is presented in Sect. 4. The case with half-integer values of the ratio of the magnetic flux to the flux quantum requires a special consideration. The corresponding analysis is presented in Sect. 5. Finally, Sect. 6 contains a summary of the work.

2 Energy-momentum tensor in a boundary-free conical space

In the presence of the external electromagnetic field with the vector potential A_μ , the dynamics of a massive spinor field ψ is governed by the Dirac equation

$$i\gamma^\mu(\nabla_\mu + ieA_\mu)\psi - m\psi = 0 , \quad (2.1)$$

with $\gamma^\mu = e_{(a)}^\mu \gamma^{(a)}$ being the Dirac matrices. Here $\gamma^{(a)}$ are the flat spacetime gamma matrices and $e_{(a)}^\mu$ is the basis tetrad. The covariant derivative operator is given by the relation

$$\nabla_\mu = \partial_\mu + \frac{1}{4}\gamma^{(a)}\gamma^{(b)}e_{(a)}^\nu e_{(b)\nu;\mu} , \quad (2.2)$$

where ";" means the standard covariant derivative for vector fields. As a background geometry, we consider a $(2+1)$ -dimensional conical spacetime with the line element

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = dt^2 - dr^2 - r^2 d\phi^2, \quad (2.3)$$

with $r \geq 0$ and $0 \leq \phi \leq \phi_0$. In $(2+1)$ -dimensional spacetime there are two inequivalent irreducible representations of the Clifford algebra. In what follows we choose the flat space Dirac matrices in the form $\gamma^{(0)} = \sigma_3$, $\gamma^{(1)} = i\sigma_1$, $\gamma^{(2)} = i\sigma_2$, where σ_l are Pauli matrices. In the second representation the gamma matrices can be taken as $\gamma^{(0)} = -\sigma_3$, $\gamma^{(1)} = -i\sigma_1$, $\gamma^{(2)} = -i\sigma_2$. The corresponding results for the second representation are obtained by changing the sign of the mass, $m \rightarrow -m$.

We assume the presence of a circular boundary with radius a on which the field obeys the MIT bag boundary condition

$$(1 + in_\mu \gamma^\mu)\psi|_{r=a} = 0 , \quad (2.4)$$

where n_μ is the outward directed normal (with respect to the region under consideration) to the boundary. We have $n_\mu = \delta_\mu^1$ and $n_\mu = -\delta_\mu^1$ for the interior and exterior regions respectively. As it will be shown below, the VEV is decomposed into the boundary-free and boundary induced parts. In this section, we shall be concerned with the VEV of the energy-momentum tensor operator for a spinor field in the boundary-free conical space. We assume the magnetic field configuration corresponding to an infinitely thin magnetic flux located at the apex of the cone. In the cylindrical coordinates of Eq. (2.3), the corresponding vector potential has the components $A_\mu = (0, 0, A)$ for $r > 0$. The z -component is related to the magnetic flux Φ by the formula $A = -\Phi/\phi_0$.

For the evaluation of the VEV of the energy-momentum tensor we use the mode-sum formula

$$\langle 0 | T_{\mu\nu} | 0 \rangle = \frac{i}{2} \sum_{\sigma} \left[\bar{\psi}_{\sigma}^{(-)}(x) \gamma_{(\mu} \nabla_{\nu)} \psi_{\sigma}^{(-)}(x) - (\nabla_{(\mu} \bar{\psi}_{\sigma}^{(-)}(x)) \gamma_{\nu)} \psi_{\sigma}^{(-)}(x) \right] . \quad (2.5)$$

where $\{\psi_{\sigma}^{(+)}, \psi_{\sigma}^{(-)}\}$ is a complete set of positive- and negative-energy solutions to the Dirac equation, $\bar{\psi} = \psi^\dagger \gamma^0$ is the Dirac adjoint and the dagger denotes Hermitian conjugation. Here σ stands for a set of quantum numbers specifying the solutions (see below). The theory of von Neumann deficiency indices leads to a one-parameter family of allowed boundary conditions in the background of an Aharonov-Bohm gauge field [37]. Here we consider a special case of boundary conditions at the cone apex, when the MIT bag boundary condition is imposed at a finite radius, which is then taken to zero. The VEVs for other boundary conditions are evaluated in a similar way. The contribution of the regular modes is the same for all boundary conditions and the results differ by the parts related to the irregular modes.

The mode functions in the boundary-free conical space are specified by the set $\sigma = (\gamma, j)$ with $0 \leq \gamma < \infty$ and $j = \pm 1/2, \pm 3/2, \dots$. The corresponding normalized negative-energy eigenspinors are given by the expression [21]

$$\psi_{(0)\gamma j}^{(-)} = \left(\gamma \frac{E+m}{2\phi_0 E} \right)^{1/2} e^{-iqj\phi + iEt} \begin{pmatrix} \frac{\gamma \epsilon_j e^{-iq\phi/2}}{E+m} J_{\beta_j + \epsilon_j}(\gamma r) \\ J_{\beta_j}(\gamma r) e^{iq\phi/2} \end{pmatrix}, \quad (2.6)$$

where $E = \sqrt{\gamma^2 + m^2}$ and $J_\nu(x)$ is the Bessel function. In Eq. (2.6) we have defined

$$\beta_j = q|j + \alpha| - \epsilon_j/2, \quad q = 2\pi/\phi_0, \quad (2.7)$$

with

$$\alpha = eA/q = -e\Phi/2\pi, \quad (2.8)$$

and

$$\epsilon_j = \begin{cases} 1, & j > -\alpha \\ -1, & j < -\alpha \end{cases}. \quad (2.9)$$

The expression for the positive-energy eigenspinor is found from Eq. (2.6) by using the relation $\psi_{\gamma j}^{(+)} = \sigma_1 \psi_{\gamma j}^{(-)*}$, where the asterisk means complex conjugate. Here we assume that the parameter α is not a half-integer. The special case of half-integer α will be considered separately in Sect. 5.

Substituting the eigenspinors (2.6) into the mode-sum (2.5), for the VEV of the energy-momentum tensor in the boundary-free geometry, $\langle 0 | T_\mu^\nu | 0 \rangle = \langle T_\mu^\nu \rangle_0$, one finds

$$\begin{aligned} \langle T_0^0 \rangle_0 &= -\frac{q}{4\pi} \sum_j \int_0^\infty d\gamma \gamma [(E-m) J_{\beta_j + \epsilon_j}^2(\gamma r) + (E+m) J_{\beta_j}^2(\gamma r)], \\ \langle T_1^1 \rangle_0 &= \frac{q}{4\pi} \sum_j \epsilon_j \int_0^\infty d\gamma \frac{\gamma^3}{E} [J_{\beta_j}(\gamma r) J'_{\beta_j + \epsilon_j}(\gamma r) - J'_{\beta_j}(\gamma r) J_{\beta_j + \epsilon_j}(\gamma r)], \\ \langle T_2^2 \rangle_0 &= \frac{q}{4\pi r} \sum_j \epsilon_j \int_0^\infty d\gamma \frac{\gamma^2}{E} (2\epsilon_j \beta_j + 1) J_{\beta_j}(\gamma r) J_{\beta_j + \epsilon_j}(\gamma r), \end{aligned} \quad (2.10)$$

where \sum_j means the summation over $j = \pm 1/2, \pm 3/2, \dots$ and the prime means the derivative with respect to the argument of the function. By using the relation

$$J'_{\beta_j}(z) = -\epsilon_j J_{\beta_j + \epsilon_j}(z) + \epsilon_j \frac{\beta_j}{z} J_{\beta_j}(z), \quad (2.11)$$

the VEVs of the energy density and radial stress may be written in the form

$$\langle T_0^0 \rangle_0 = -A_0(r) + m \langle \bar{\psi} \psi \rangle_0, \quad \langle T_1^1 \rangle_0 = A_0(r) - \langle T_2^2 \rangle_0, \quad (2.12)$$

with $\langle \bar{\psi} \psi \rangle_0$ being the fermionic condensate (see Ref. [22]) and

$$A_0(r) = \frac{q}{4\pi} \sum_j \int_0^\infty d\gamma \frac{\gamma^3}{E} [J_{\beta_j}^2(\gamma r) + J_{\beta_j + \epsilon_j}^2(\gamma r)]. \quad (2.13)$$

From these expressions the trace relation $\langle T_k^k \rangle_0 = m \langle \bar{\psi} \psi \rangle_0$ is explicitly seen. Another relation between the separate components is a consequence of the covariant conservation equation for the energy-momentum tensor. For the geometry at hand the latter is reduced to a single equation: $\partial_r (r \langle T_1^1 \rangle_0) = \langle T_2^2 \rangle_0$.

If we present the parameter α related to the magnetic flux as

$$\alpha = \alpha_0 + n_0, \quad |\alpha_0| < 1/2, \quad (2.14)$$

with n_0 being an integer number, it can be seen that the VEVs do not depend on n_0 . Hence, we conclude that the VEV of the energy-momentum tensor depends on α_0 alone. The VEV is an even function of this parameter (note that the same is the case for the fermionic condensate, whereas the VEV of the fermionic current is an odd function of α_0).

The expressions (2.10) are divergent and need to be regularized. We introduce a cutoff function $e^{-s\gamma^2}$ with the cutoff parameter $s > 0$. At the end of calculations the limit $s \rightarrow 0$ is taken. First we consider the azimuthal stress. From Eq. (2.10), the corresponding regularized VEV can be written in the form

$$\langle T_2^2 \rangle_{0,\text{reg}} = \frac{q}{8\pi r^2} \sum_j (2\beta_j + \epsilon_j) (2\beta_j - \epsilon_j r \partial_r) \int_0^\infty d\gamma \frac{\gamma e^{-s\gamma^2}}{\sqrt{\gamma^2 + m^2}} J_{\beta_j}^2(\gamma r). \quad (2.15)$$

By using the relation

$$\frac{1}{\sqrt{\gamma^2 + m^2}} = \frac{2}{\sqrt{\pi}} \int_0^\infty dt e^{-(\gamma^2 + m^2)t^2}, \quad (2.16)$$

and changing the order of integrations, the γ -integral is performed explicitly (see Ref. [38]) with the result

$$\begin{aligned} \langle T_2^2 \rangle_{0,\text{reg}} &= \frac{q}{8\pi r^2} \frac{e^{m^2 s}}{\sqrt{2\pi}} \sum_j (2\beta_j + \epsilon_j) (2\beta_j - \epsilon_j r \partial_r) \\ &\times \int_0^{1/(2s)} dy \frac{y^{-1/2} I_{\beta_j}(r^2 y)}{\sqrt{1 - 2ys}} e^{-m^2/(2y) - r^2 y}, \end{aligned} \quad (2.17)$$

where $I_{\beta_j}(x)$ is the modified Bessel function. By using the properties of the modified Bessel function, Eq. (2.17) can also be written in the form

$$\langle T_2^2 \rangle_{0,\text{reg}} = \frac{q e^{m^2 s}}{2(2\pi)^{3/2}} \partial_r r \int_0^{1/(2s)} dy \frac{y^{1/2} e^{-m^2/(2y) - r^2 y}}{\sqrt{1 - 2ys}} \sum_j [I_{\beta_j}(r^2 y) + I_{\beta_j + \epsilon_j}(r^2 y)]. \quad (2.18)$$

In order to find the representation for regularized VEVs of the energy density and radial stress we need to consider the expression for $A_0(r)$ in Eq. (2.13) regularized with the cutoff function $e^{-s\gamma^2}$. The regularized expression can be presented in the form

$$A_{0,\text{reg}}(r) = -\frac{q}{4\pi} \sum_j \partial_s \int_0^\infty d\gamma \frac{\gamma e^{-s\gamma^2}}{\sqrt{\gamma^2 + m^2}} [J_{\beta_j}^2(\gamma r) + J_{\beta_j + \epsilon_j}^2(\gamma r)]. \quad (2.19)$$

The parts with separate terms in the square brackets are evaluated in a way similar to that we used for Eq. (2.15). As a result we find

$$A_{0,\text{reg}}(r) = \frac{q e^{m^2 s}}{(2\pi)^{3/2}} \sum_j \frac{\partial}{\partial r^2} r^2 \int_0^{1/(2s)} dy \frac{y^{1/2} e^{-m^2/(2y) - r^2 y}}{\sqrt{1 - 2ys}} [I_{\beta_j}(r^2 y) + I_{\beta_j + \epsilon_j}(r^2 y)]. \quad (2.20)$$

From here, in the combination with Eq. (2.12), for the regularized radial stress we obtain the expression

$$\langle T_1^1 \rangle_{0,\text{reg}} = \frac{q e^{m^2 s}}{2(2\pi)^{3/2}} \int_0^{1/(2s)} dy \frac{y^{1/2} e^{-m^2/(2y) - r^2 y}}{\sqrt{1 - 2ys}} \sum_j [I_{\beta_j}(r^2 y) + I_{\beta_j + \epsilon_j}(r^2 y)]. \quad (2.21)$$

For the energy density and the azimuthal stress we have:

$$\begin{aligned}\langle T_0^0 \rangle_{0,\text{reg}} &= -(2 + r\partial_r) \langle T_1^1 \rangle_{0,\text{reg}} + m \langle \bar{\psi}\psi \rangle_{0,\text{reg}}, \\ \langle T_2^2 \rangle_{0,\text{reg}} &= (1 + r\partial_r) \langle T_1^1 \rangle_{0,\text{reg}}.\end{aligned}\tag{2.22}$$

Note that by the second relation we explicitly checked the covariant continuity equation for the regularized VEVs.

As the fermionic condensate has been considered in Ref. [22], in accordance with Eq. (2.22) we need to consider the radial stress only. The corresponding regularized VEV is expressed in terms of the series

$$\mathcal{I}(q, \alpha, z) = \sum_j I_{\beta_j}(z).\tag{2.23}$$

If we present the parameter α in the form (2.14), it is easily seen the independence of the series on n_0 : $\mathcal{I}(q, \alpha, z) = \mathcal{I}(q, \alpha_0, z)$. For the second series appearing in the expressions for the regularized VEVs we have

$$\sum_j I_{\beta_j + \epsilon_j}(z) = \mathcal{I}(q, -\alpha_0, z).\tag{2.24}$$

With the notation (2.23), the regularized VEV of the radial stress is written in the form

$$\langle T_1^1 \rangle_{0,\text{reg}} = \frac{qe^{m^2 s}}{2(2\pi)^{3/2}} \int_0^{1/(2s)} dy \frac{y^{1/2} e^{-m^2/(2y) - r^2 y}}{\sqrt{1 - 2ys}} \sum_{j=\pm 1} \mathcal{I}(q, j\alpha_0, r^2 y).\tag{2.25}$$

For $2p < q < 2p + 2$, with p being an integer, we use the representation [21]

$$\mathcal{I}(q, \alpha_0, z) = \frac{e^z}{q} + \mathcal{J}(q, \alpha_0, z),\tag{2.26}$$

with the notation

$$\begin{aligned}\mathcal{J}(q, \alpha_0, z) &= -\frac{1}{\pi} \int_0^\infty dy \frac{e^{-z \cosh y} f(q, \alpha_0, y)}{\cosh(qy) - \cos(q\pi)} \\ &\quad + \frac{2}{q} \sum_{l=1}^p (-1)^l \cos[2\pi l(\alpha_0 - 1/2q)] e^{z \cos(2\pi l/q)}.\end{aligned}\tag{2.27}$$

The function in the integrand is defined by the expression

$$\begin{aligned}f(q, \alpha_0, y) &= \cos[q\pi(1/2 - \alpha_0)] \cosh[(q\alpha_0 + q/2 - 1/2)y] \\ &\quad - \cos[q\pi(1/2 + \alpha_0)] \cosh[(q\alpha_0 - q/2 - 1/2)y].\end{aligned}\tag{2.28}$$

In the case $q = 2p$, the term

$$-(-1)^{q/2} \frac{e^{-z}}{q} \sin(q\pi\alpha_0),\tag{2.29}$$

should be added to the right-hand side of Eq. (2.27). For $1 \leq q < 2$, the last term on the right-hand side of Eq. (2.27) is absent.

Substituting Eq. (2.26) into the right-hand side of Eq. (2.25), we can see that the only divergent contribution comes from the term e^z/q . This contribution does not depend on the opening angle of the cone and on the magnetic flux. It coincides with the corresponding quantity in the Minkowski spacetime, in the absence of the magnetic flux. Subtracting the Minkowskian part and taking the limit $s \rightarrow 0$, after the explicit integration over y , we get the expression for the renormalized radial stress, $\langle T_1^1 \rangle_{0,\text{ren}}$. The corresponding expressions for the energy density

and the azimuthal stress are found from Eq. (2.22), by using the expression for $\langle \bar{\psi}\psi \rangle_{0,\text{ren}}$ from Ref. [22]. In this way, one finds the following formula (no summation over i)

$$\begin{aligned} \langle T_i^i \rangle_{0,\text{ren}} &= \frac{m^3}{\pi} \left[\sum_{l=1}^p (-1)^l \cos(\pi l/q) \cos(2\pi l\alpha_0) F_i^{(s)}(2mr s_l) \right. \\ &\quad \left. - \frac{q}{2\pi} \int_0^\infty dy \frac{\sum_{\delta=\pm 1} f(q, \delta\alpha_0, 2y) F_i^{(s)}(2mr \cosh(y))}{\cosh(2qy) - \cos(q\pi)} \right], \end{aligned} \quad (2.30)$$

where p is an integer defined by $2p \leq q < 2p + 2$, and

$$s_l = \sin(\pi l/q). \quad (2.31)$$

In Eq. (2.30), we have defined the functions

$$\begin{aligned} F_0^{(s)}(u) &= F_1^{(s)}(u) = \frac{e^{-u}}{u^3} (u + 1), \\ F_2^{(s)}(u) &= -\frac{e^{-u}}{u^3} (u^2 + 2u + 2). \end{aligned} \quad (2.32)$$

Note that for the function in the integrand one has:

$$\sum_{\delta=\pm 1} f(q, \delta\alpha_0, 2y) = -2 \sinh(y) \sum_{\delta=\pm 1} \cos[q\pi(1/2 + \delta\alpha_0)] \sinh[q(1 - 2\delta\alpha_0)y]. \quad (2.33)$$

As it is seen, the radial stress is equal to the energy density: $\langle T_0^0 \rangle_{0,\text{ren}} = \langle T_1^1 \rangle_{0,\text{ren}}$.

For a massless field the corresponding energy density is directly obtained from Eq. (2.30):

$$\begin{aligned} \langle T_0^0 \rangle_{0,\text{ren}} &= \frac{1}{8\pi r^3} \left[\sum_{l=1}^p \frac{(-1)^l}{s_l^3} \cos(\pi l/q) \cos(2\pi l\alpha_0) \right. \\ &\quad \left. - \frac{q}{2\pi} \int_0^\infty dy \frac{\sum_{j=\pm 1} f(q, j\alpha_0, 2y)}{\cosh(2qy) - \cos(q\pi)} \frac{1}{\cosh^3 y} \right]. \end{aligned} \quad (2.34)$$

For the radial and azimuthal stresses one has

$$\langle T_1^1 \rangle_{0,\text{ren}} = -\frac{1}{2} \langle T_2^2 \rangle_{0,\text{ren}} = \langle T_0^0 \rangle_{0,\text{ren}}. \quad (2.35)$$

Of course, in the massless case the energy-momentum tensor is traceless. In Fig. 1 we plot the renormalized energy density for a massless field as a function of the parameter α_0 for separate values of the parameter q (numbers near the curves).

For a massive field, the expression in the right-hand side of Eq. (2.34) gives the leading term in the corresponding asymptotic expansion for small distances from the string, $mr \ll 1$. At distances larger than the Compton wavelength of the spinor particle, $mr \gg 1$, the VEVs are suppressed by the factor e^{-2mr} for $1 \leq q \leq 2$ and by the factor $e^{-2mr \sin(\pi/q)}$ for $q > 2$. In the latter case the dominant contribution comes from the first term in the right-hand side of Eq. (2.30):

$$\begin{aligned} \langle T_i^i \rangle_{0,\text{ren}} &\approx -\frac{m^2}{2\pi r} \cot(\pi/q) \cos(2\pi\alpha_0) \\ &\quad \times e^{-2mr \sin(\pi/q)} \begin{cases} 1/[2mr \sin(\pi/q)], & i = 0, 1 \\ -1, & i = 2 \end{cases}, \end{aligned} \quad (2.36)$$

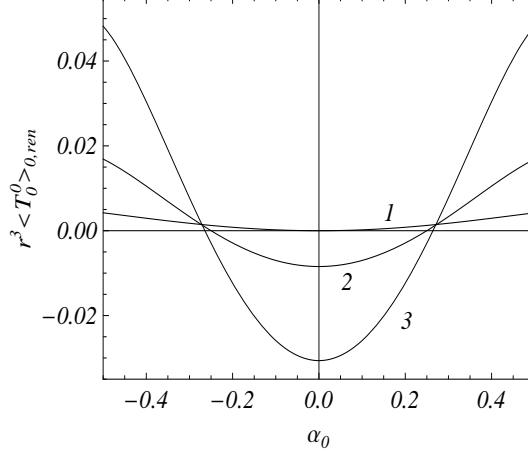


Figure 1: Energy density for a massless fermionic field as a function of the parameter α_0 for separate values of the parameter q (numbers near the curves). The vacuum stresses are related to the energy density by Eq. (2.35).

for $mr \gg 1$.

For integer q and for the parameter α given by the special value

$$\alpha = 1/2q - 1/2, \quad (2.37)$$

the expression (2.30) for the VEVs take the form (no summation)

$$\langle T_i^i \rangle_{0,\text{ren}} = \frac{m^3}{2\pi} \sum_{l=1}^{q-1} \cos^2(\pi l/q) F_i^{(s)}(2mr s_l), \quad (2.38)$$

Note that, in this case, the renormalized VEV vanishes in a conical space with $q = 2$. For $q \geq 3$ the energy density is positive.

The energy density diverges on the string as $1/r^3$ and, as a result, the integrated energy diverges as well. We can evaluate the total vacuum energy in the region $r_0 \leq r < \infty$ by using the energy density given above: $E_{0,r \geq r_0} = \phi_0 \int_{r_0}^{\infty} dr r \langle T_0^0 \rangle_{0,\text{ren}}$. Performing the radial integration, we find

$$E_{0,r \geq r_0} = \frac{1}{4qr_0} \left[\sum_{l=1}^p \frac{(-1)^l}{s_l^3} \cos(\pi l/q) \cos(2\pi l \alpha_0) e^{-2mr_0 s_l} - \frac{q}{2\pi} \int_0^{\infty} dy \frac{\sum_{j=\pm 1} f(q, j\alpha_0, 2y)}{\cosh(2qy) - \cos(q\pi)} \frac{e^{-2mr_0 \cosh y}}{\cosh^3 y} \right]. \quad (2.39)$$

For a massless field we have a simple relation $E_{0,r \geq r_0} = \phi_0 r_0^2 \langle T_0^0 \rangle_{0,\text{ren}}|_{r=r_0}$, which could also be directly obtained from Eq. (2.34).

In the special case when the magnetic flux is absent we have $\alpha_0 = 0$ and the general formula simplifies to (no summation over i)

$$\begin{aligned} \langle T_i^i \rangle_{0,\text{ren}} &= \frac{m^3}{\pi} \left[\sum_{l=1}^p (-1)^l \cos(\pi l/q) F_i^{(s)}(2mr s_l) \right. \\ &\quad \left. + \frac{2q}{\pi} \cos\left(\frac{q\pi}{2}\right) \int_0^{\infty} dy \frac{F_i^{(s)}(2mr \cosh y) \sinh(qy) \sinh y}{\cosh(2qy) - \cos(q\pi)} \right]. \end{aligned} \quad (2.40)$$

In this case, the VEV is only a consequence of the conical structure of the space. For odd values of the parameter q the second term in the square brackets vanishes and for the VEV we have the simple formula (no summation over i)

$$\langle T_i^i \rangle_{0,\text{ren}} = \frac{m^3}{\pi} \sum_{l=1}^p (-1)^l \cos(\pi l/q) F_i^{(s)}(2mr s_l). \quad (2.41)$$

Another special case corresponds to the magnetic flux in background of Minkowski spacetime. In this case, taking $q = 1$, from the general formulas we find

$$\langle T_i^i \rangle_{0,\text{ren}} = \frac{m^3}{\pi^2} \sin(\pi \alpha_0) \int_0^\infty dy \tanh(y) \sinh(2\alpha_0 y) F_i^{(s)}(2mr \cosh(y)), \quad (2.42)$$

and the corresponding energy density is positive for $\alpha_0 \neq 0$.

An alternative expression for the VEV of the energy-momentum tensor is obtained by using the formula [21]

$$\begin{aligned} \mathcal{I}(q, \alpha_0, x) &= \frac{2}{q} \int_0^\infty dz I_z(x) + A(q, \alpha_0, x) \\ &\quad - \frac{4}{\pi q} \int_0^\infty dz \operatorname{Re} \left[\frac{\sinh(z\pi) K_{iz}(x)}{e^{2\pi(z+i|q\alpha_0-1/2|)/q} + 1} \right], \end{aligned} \quad (2.43)$$

with $K_\nu(x)$ being the modified Bessel function. In Eq. (2.43), $A(q, \alpha_0, x) = 0$ for $|\alpha_0 - 1/2q| \leq 1/2$, and

$$A(q, \alpha_0, x) = \frac{2}{\pi} \sin[\pi(|q\alpha_0 - 1/2| - q/2)] K_{|q\alpha_0-1/2|-q/2}(x), \quad (2.44)$$

for $1/2 < |\alpha_0 - 1/2q| < 1$. Substituting the representation (2.43) into the expressions for the regularized VEVs, we see that the part with the first term on the right-hand side of Eq. (2.43) does not depend on the opening angle of the cone and on the magnetic flux. This term coincides with the corresponding result in Minkowski bulk when the magnetic flux is absent. Hence, it should be subtracted in the renormalization procedure. In the remaining part the limit $s \rightarrow 0$ can be taken directly.

In a same way, we can consider a more general problem where the spinor field obeys quasiperiodic boundary condition along the azimuthal direction

$$\psi(t, r, \phi + \phi_0) = e^{2\pi i \chi} \psi(t, r, \phi), \quad (2.45)$$

with a constant parameter χ , $|\chi| \leq 1/2$. For this problem, the exponential factor in the expression for the mode functions (2.6) has the form $e^{-iq(n+\chi)\phi + iEt}$. The corresponding expression for the mode functions is obtained from that given above with the parameter α defined by

$$\alpha = \chi - e\Phi/2\pi. \quad (2.46)$$

For the case of a field with periodicity condition (2.45), the expressions of the renormalized VEVs for the energy density and stresses are given by the previous formulas where now the parameter α is defined as in Eq. (2.46).

In general, the fermionic modes in the background of the magnetic vortex are divided into two classes, regular and irregular (square integrable) ones. For given q and α , the irregular mode corresponds to the value of j for which $q|j + \alpha| < 1/2$. If we present the parameter α in the form (2.14), then the irregular mode is present if $|\alpha_0| > (1 - 1/q)/2$. This mode corresponds to $j = -n_0 - \operatorname{sgn}(\alpha_0)/2$. Note that, in a conical space, under the condition $|\alpha_0| \leq (1 - 1/q)/2$, there

are no square integrable irregular modes. As we have already mentioned, there is a one-parameter family of allowed boundary conditions for irregular modes. These modes are parameterized by the angle θ , $0 \leq \theta < 2\pi$ (see Ref. [37]). For $|\alpha_0| < 1/2$, the boundary condition, used in deriving mode functions (2.6), corresponds to $\theta = 3\pi/2$. If α is a half-integer, the irregular mode corresponds to $j = -\alpha$ and for the corresponding boundary condition one has $\theta = 0$. Note that in both cases there are no bound states.

3 Energy-momentum tensor inside a circular boundary

We turn to the investigation of the effect of a circular boundary on the VEV of the energy-momentum tensor for a spinor field. We assume that on the circle the field obeys the MIT bag boundary condition (2.4). First we consider the region inside the boundary. In this region the negative-energy eigenspinors are given by the expression [21]

$$\psi_{\gamma j}^{(-)} = \varphi_0 e^{-iqj\phi + iEt} \begin{pmatrix} \frac{\epsilon_j \gamma e^{-iq\phi/2}}{E+m} J_{\beta_j + \epsilon_j}(\gamma r) \\ e^{iq\phi/2} J_{\beta_j}(\gamma r) \end{pmatrix}, \quad (3.1)$$

with the same notations as in Eq. (2.6). From the boundary condition at $r = a$ it follows that the allowed values of γ are solutions of the equation

$$J_{\beta_j}(\gamma a) - \frac{\gamma \epsilon_j J_{\beta_j + \epsilon_j}(\gamma a)}{m + \sqrt{\gamma^2 + m^2}} = 0. \quad (3.2)$$

For a given β_j , Eq. (3.2) has an infinite number of solutions which we denote by $\gamma a = \gamma_{\beta_j, l}$, $l = 1, 2, \dots$. The normalization coefficient in Eq. (3.1) is given by the expression

$$\varphi_0^2 = \frac{\gamma T_{\beta_j}(\gamma a)}{2\phi_0 a} \frac{m + E}{E}, \quad (3.3)$$

with the notation

$$T_{\beta_j}(y) = \frac{y}{J_{\beta_j}^2(y)} \left[y^2 + (\mu - \epsilon_j \beta_j) \left(\mu + \sqrt{y^2 + \mu^2} \right) - \frac{y^2}{2\sqrt{y^2 + \mu^2}} \right]^{-1}, \quad (3.4)$$

and $\mu = ma$.

Substituting the mode functions (3.1) into Eq. (2.5) with $\sum_\sigma = \sum_j \sum_{l=1}^\infty$, for the VEVs of the separate components we find

$$\begin{aligned} \langle T_0^0 \rangle &= -\frac{q}{4\pi a} \sum_j \sum_{l=1}^\infty \gamma T_{\beta_j}(\gamma a) \left[(E - m) J_{\beta_j + \epsilon_j}^2(\gamma r) + (E + m) J_{\beta_j}^2(\gamma r) \right], \\ \langle T_1^1 \rangle &= -\frac{q}{4\pi a} \sum_j \sum_{l=1}^\infty \epsilon_j \frac{\gamma^3}{E} T_{\beta_j}(\gamma a) [J'_{\beta_j}(\gamma r) J_{\beta_j + \epsilon_j}(\gamma r) - J'_{\beta_j + \epsilon_j}(\gamma r) J_{\beta_j}(\gamma r)], \\ \langle T_2^2 \rangle &= \frac{q}{4\pi a} \sum_j \sum_{l=1}^\infty \frac{\gamma^3}{E} T_{\beta_j}(\gamma a) \frac{2\beta_j + \epsilon_j}{\gamma r} J_{\beta_j}(\gamma r) J_{\beta_j + \epsilon_j}(\gamma r), \end{aligned} \quad (3.5)$$

with $\gamma = \gamma_{\beta_j, l}/a$. Here we assume that a cutoff function is introduced without explicitly writing it. The specific form of this function is not important for the discussion below.

For the summation of the series over l in Eq. (3.5) we use the summation formula (see Refs. [39, 40])

$$\sum_{l=1}^{\infty} f(\gamma_{\beta_j, l}) T_{\beta}(\gamma_{\beta_j, l}) = \int_0^{\infty} dx f(x) - \frac{1}{\pi} \int_0^{\infty} dx \times [e^{-\beta_j \pi i} f(x e^{\pi i/2}) L_{\beta_j}^{(+)}(x) + e^{\beta_j \pi i} f(x e^{-\pi i/2}) L_{\beta_j}^{(+)*}(x)], \quad (3.6)$$

where

$$L_{\beta_j}^{(+)}(x) = \frac{K_{\beta_j}^{(+)}(x)}{I_{\beta_j}^{(+)}(x)}, \quad (3.7)$$

and the asterisk means complex conjugate. In Eq. (3.7), for a given function $F(x)$, we use the notation

$$F^{(+)}(x) = \begin{cases} xF'(x) + (\mu + \sqrt{\mu^2 - x^2} - \epsilon_j \beta_j) F(x), & x < \mu, \\ xF'(x) + (\mu + i\sqrt{x^2 - \mu^2} - \epsilon_j \beta_j) F(x), & x \geq \mu. \end{cases} \quad (3.8)$$

Note that for $x < \mu$ one has $F^{(+)*}(x) = F^{(+)}(x)$. By using the properties of the modified Bessel functions, the function $L_{\beta_j}^{(+)}(x)$ can be presented in the form

$$L_{\beta_j}^{(+)}(x) = \frac{W_{\beta_j, \beta_j + \epsilon_j}^{(+)}(x) + i\sqrt{1 - \mu^2/x^2}}{U_{\beta_j, \beta_j + \epsilon_j}^{(I)}(x)}, \quad (3.9)$$

with the notations defined by

$$\begin{aligned} W_{\nu, \sigma}^{(\pm)}(x) &= x [I_{\nu}(x) K_{\sigma}(x) - I_{\sigma}(x) K_{\nu}(x)] \\ &\quad \pm \mu [I_{\sigma}(x) K_{\nu}(x) - I_{\nu}(x) K_{\sigma}(x)], \\ U_{\nu, \sigma}^{(I)}(x) &= x [I_{\nu}^2(x) + I_{\sigma}^2(x)] + 2\mu I_{\nu}(x) I_{\sigma}(x). \end{aligned} \quad (3.10)$$

The function $W_{\nu, \sigma}^{(-)}(x)$ will appear in the expressions for the VEV in the exterior region (see Sect. 4).

Applying to the series over l in Eq. (3.5) the summation formula, it can be seen that the terms in the VEVs corresponding to the first integral in the right-hand side of Eq. (3.6) coincide with the corresponding VEVs in a boundary-free conical space. As a result, after the application of formula (3.6), the VEV of the energy-momentum tensor is presented in the decomposed form

$$\langle T_k^i \rangle = \langle T_k^i \rangle_{0, \text{ren}} + \langle T_k^i \rangle_{\text{b}}, \quad (3.11)$$

with $\langle T_k^i \rangle_{\text{b}}$ being the part induced by the circular boundary. For the functions $f(x)$ corresponding to Eq. (3.5), in the second term on the right-hand side of Eq. (3.6), the part of the integral over the region $(0, \mu)$ vanishes. As a result, the boundary-induced contributions in the interior region are given by the expressions

$$\begin{aligned} \langle T_0^0 \rangle_{\text{b}} &= -\frac{q}{2\pi^2} \sum_j \int_m^{\infty} dx \frac{\sqrt{x^2 - m^2}}{U_{\beta_j, \beta_j + \epsilon_j}^{(I)}(ax)} \{m[I_{\beta_j}^2(rx) + I_{\beta_j + \epsilon_j}^2(rx)] \\ &\quad + x[I_{\beta_j}^2(rx) - I_{\beta_j + \epsilon_j}^2(rx)] W_{\beta_j, \beta_j + \epsilon_j}^{(+)}(ax)\}, \\ \langle T_1^1 \rangle_{\text{b}} &= -\frac{q}{2\pi^2} \sum_j \int_m^{\infty} dx \frac{x^3 W_{\beta_j, \beta_j + \epsilon_j}^{(+)}(ax)}{\sqrt{x^2 - m^2}} \frac{I'_{\beta_j}(rx) I_{\beta_j + \epsilon_j}(rx) - I_{\beta_j}(rx) I'_{\beta_j + \epsilon_j}(rx)}{U_{\beta_j, \beta_j + \epsilon_j}^{(I)}(ax)}, \\ \langle T_2^2 \rangle_{\text{b}} &= \frac{q}{2\pi^2 r} \sum_j (2\epsilon_j \beta_j + 1) \int_m^{\infty} dx \frac{x^2 W_{\beta_j, \beta_j + \epsilon_j}^{(+)}(ax)}{\sqrt{x^2 - m^2}} \frac{I_{\beta_j}(rx) I_{\beta_j + \epsilon_j}(rx)}{U_{\beta_j, \beta_j + \epsilon_j}^{(I)}(ax)}. \end{aligned} \quad (3.12)$$

For points away from the circular boundary and the cone apex, the boundary-induced contributions, given by Eq. (3.12), are finite and the renormalization is reduced to that for the boundary-free geometry. The latter we have discussed in the previous section.

Under the change $\alpha \rightarrow -\alpha$, $j \rightarrow -j$, we have $\beta_j \rightarrow \beta_j + \epsilon_j$, $\beta_j + \epsilon_j \rightarrow \beta_j$. From here it follows that, under this change, the functions $W_{\beta_j, \beta_j + \epsilon_j}^{(+)}(ax)$ and $U_{\beta_j, \beta_j + \epsilon_j}^{(I)}(ax)$ are odd and even functions, respectively. Now, from Eq. (3.12) we see that the boundary-induced parts in the components of the energy-momentum tensor are even functions of α . They are periodic functions of the parameter α with the period equal to 1. Consequently, if we present this parameter in the form (2.14) with n_0 being an integer, then the VEV of the energy-momentum tensor depends on α_0 alone. Note that, by using the recurrence relations for the modified Bessel function, the radial stress can also be written in the form

$$\begin{aligned} \langle T_1^1 \rangle_b &= -\frac{q}{2\pi^2} \sum_j \int_m^\infty dx \frac{x^3 W_{\beta_j, \beta_j + \epsilon_j}^{(+)}(ax)}{\sqrt{x^2 - m^2}} \\ &\quad \times \frac{I_{\beta_j + \epsilon_j}^2(rx) - I_{\beta_j}^2(rx)}{U_{\beta_j, \beta_j + \epsilon_j}^{(I)}(ax)} - \langle T_2^2 \rangle_b. \end{aligned} \quad (3.13)$$

Now, it is easy to explicitly check that the trace relation $\langle T_i^i \rangle_b = m \langle \bar{\psi} \psi \rangle_b$ is satisfied.

In the case of a massless field the expressions for the boundary-induced parts in the VEVs take the form

$$\begin{aligned} \langle T_0^0 \rangle_b &= -\frac{q}{2\pi^2 a^3} \sum_j \int_0^\infty dx x^2 V_{\beta_j, \beta_j + \epsilon_j}^{(I)}(x) [I_{\beta_j}^2(xr/a) - I_{\beta_j + \epsilon_j}^2(xr/a)], \\ \langle T_2^2 \rangle_b &= \frac{q}{2\pi^2 a^2 r} \sum_j (2\epsilon_j \beta_j + 1) \int_0^\infty dx x V_{\beta_j, \beta_j + \epsilon_j}^{(I)}(x) I_{\beta_j}^2(xr/a) I_{\beta_j + \epsilon_j}^2(xr/a), \end{aligned} \quad (3.14)$$

where we have defined

$$V_{\nu, \sigma}^{(I)}(x) = \frac{I_\nu(x) K_\sigma(x) - I_\sigma(x) K_\nu(x)}{I_\nu^2(x) + I_\sigma^2(x)}, \quad (3.15)$$

and for the radial stress we have $\langle T_1^1 \rangle_b = -\langle T_0^0 \rangle_b - \langle T_2^2 \rangle_b$.

Let us consider asymptotic behavior of the VEV for the energy-momentum tensor near the cone apex and near the boundary. In the limit $r \rightarrow 0$ we use the expansion for the modified Bessel functions for small values of the argument. Writing the parameter α in the form (2.14), it is seen that the dominant contribution comes from the term with $j = -n_0 - 1/2$ for $\alpha_0 > 0$ and from the term $j = -n_0 + 1/2$ for $\alpha_0 < 0$. The leading terms in the expansions over r/a are given by the expressions

$$\begin{aligned} \langle T_0^0 \rangle_b &\approx \frac{q a^{-3}}{2^{2q_\alpha} \pi^2} \frac{(r/a)^{2q_\alpha - 1}}{\Gamma^2(q_\alpha + 1/2)} \int_\mu^\infty dx \frac{x^{2q_\alpha} \sqrt{x^2 - \mu^2}}{U_{q_\alpha + 1/2, q_\alpha - 1/2}^{(I)}(x)} [W_{q_\alpha + 1/2, q_\alpha - 1/2}^{(+)}(x) - \mu/x], \\ \langle T_1^1 \rangle_b &\approx -\frac{q a^{-3}}{2^{2q_\alpha} \pi^2} \frac{(r/a)^{2q_\alpha - 1}}{(2q_\alpha + 1) \Gamma^2(q_\alpha + 1/2)} \int_\mu^\infty dx \frac{x^{2q_\alpha + 2}}{\sqrt{x^2 - \mu^2}} \frac{W_{q_\alpha + 1/2, q_\alpha - 1/2}^{(+)}(x)}{U_{q_\alpha + 1/2, q_\alpha - 1/2}^{(I)}(x)}, \end{aligned} \quad (3.16)$$

where

$$q_\alpha = q(1/2 - |\alpha_0|). \quad (3.17)$$

For the azimuthal stress one has $\langle T_2^2 \rangle_b = 2q_\alpha \langle T_1^1 \rangle_b$. For $\alpha_0 = 0$ the dominant contribution comes from the terms $j = -n_0 \pm 1/2$ and the corresponding asymptotics are obtained from Eq. (3.16)

taking $q_\alpha = q/2$ with an additional factor 2. As it is seen from Eq. (3.16), the boundary induced VEVs vanish on the cone apex for $|\alpha_0| < (1 - 1/q)/2$ and diverge when $|\alpha_0| > (1 - 1/q)/2$. In particular, the VEVs diverge for a magnetic flux in background of Minkowski spacetime.

For points near the boundary, the dominant contribution to the VEVs come from large values of j . Introducing in Eq. (3.12) a new integration variable $x = \beta_j y$, we use the uniform asymptotic expansions for the modified Bessel functions [41]. From these expansions it follows that to the leading order one has

$$\begin{aligned} I_{\beta_j}^2(\beta_j z) - I_{\beta_j + \epsilon_j}^2(\beta_j z) &\sim \frac{e^{2\beta_j \eta(z)}}{\pi \beta_j z^2} [\epsilon_j - t(z)], \\ I_{\beta_j}^2(\beta_j z) + I_{\beta_j + \epsilon_j}^2(\beta_j z) &\sim \frac{e^{2\beta_j \eta(z)}}{\pi \beta_j z^2} [1/t(z) - \epsilon_j], \end{aligned} \quad (3.18)$$

and

$$K_{\beta_j}(\beta_j z) I_{\beta_j}(\beta_j z) - K_{\beta_j + \epsilon_j}(\beta_j z) I_{\beta_j + \epsilon_j}(\beta_j z) \sim \frac{\epsilon_j t^3(z)}{2\beta_j^2}, \quad (3.19)$$

with the standard notations $t(z) = 1/\sqrt{1+z^2}$,

$$\eta(z) = \sqrt{1+z^2} + \ln \left(\frac{z}{1 + \sqrt{1+z^2}} \right). \quad (3.20)$$

With the help of these expressions, for the leading terms in the asymptotic expansions over the distance from the boundary one gets

$$\begin{aligned} \langle T_0^0 \rangle_b &\approx -\frac{1/8 + \mu}{16\pi a(a-r)^2}, \\ \langle T_2^2 \rangle_b &\approx \frac{a}{a-r} \langle T_1^1 \rangle_b \approx \frac{1/8 - \mu}{16\pi a(a-r)^2}. \end{aligned} \quad (3.21)$$

As it is seen, near the boundary the energy density is negative, whereas the signs of the stresses depend on the mass.

Now let us consider the limiting case when r is fixed and the radius of the circle is large. For a massive field, assuming $ma \gg 1$, we see that the dominant contribution to the VEVs (3.12) comes from the region near the lower limit of the integration. To the leading order we have:

$$\begin{aligned} \langle T_0^0 \rangle_b &\approx -\frac{qm^3 e^{-2ma}}{16\sqrt{\pi}(ma)^{3/2}} \sum_j [I_{\beta_j}^2(rm) + I_{\beta_j + \epsilon_j}^2(rm)], \\ \langle T_1^1 \rangle_b &\approx \frac{qm^3 e^{-2ma}}{16\sqrt{\pi}(ma)^{3/2}} \sum_j (2\epsilon_j \beta_j + 1) [I'_{\beta_j}(rm) I_{\beta_j + \epsilon_j}(rm) - I_{\beta_j}(rm) I'_{\beta_j + \epsilon_j}(rm)], \\ \langle T_2^2 \rangle_b &\approx -\frac{qm^3 e^{-2ma}}{16\sqrt{\pi}(ma)^{3/2}} \sum_j \frac{(2\epsilon_j \beta_j + 1)^2}{rm} I_{\beta_j}(rm) I_{\beta_j + \epsilon_j}(rm). \end{aligned} \quad (3.22)$$

In this case the boundary induced VEVs are exponentially suppressed. For a massless field and for large values of the circle radius, the corresponding behavior is obtained from Eq. (3.16) taking $\mu = 0$. In this case the decay of the VEVs is of power-law (no summation): $\langle T_i^i \rangle_b \sim 1/a^{2(q_\alpha+1)}$.

In Fig. 2 we display the boundary induced parts in the VEV of the energy density (full curves) and azimuthal stress (dashed curves) as functions of the radial coordinate for separate values of the parameter q (numbers near the curves). The left and right panels are plotted for

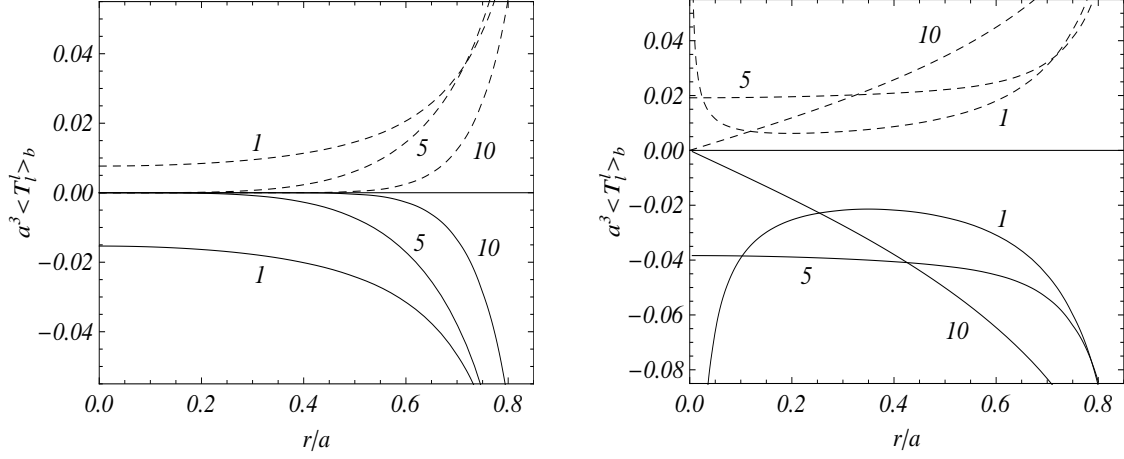


Figure 2: Boundary-induced parts in the VEV of the energy density (full curves) and azimuthal stress (dashed curves) as functions of the radial coordinate for separate values of the parameter q (numbers near the curves). The left and right panels are plotted for $\alpha_0 = 0$ and $\alpha_0 = 0.4$, respectively.

$\alpha_0 = 0$ and $\alpha_0 = 0.4$, respectively. In the second case, for $q = 5, 10$ there are no irregular modes and the VEVs are finite on the apex.

The VEVs of the vacuum energy density and the azimuthal stress for a massless field are plotted in Fig. 3 as functions of the parameter α_0 for fixed value of the radial coordinate corresponding to $r/a = 0.5$. The numbers near the curves are the values of the parameter q .

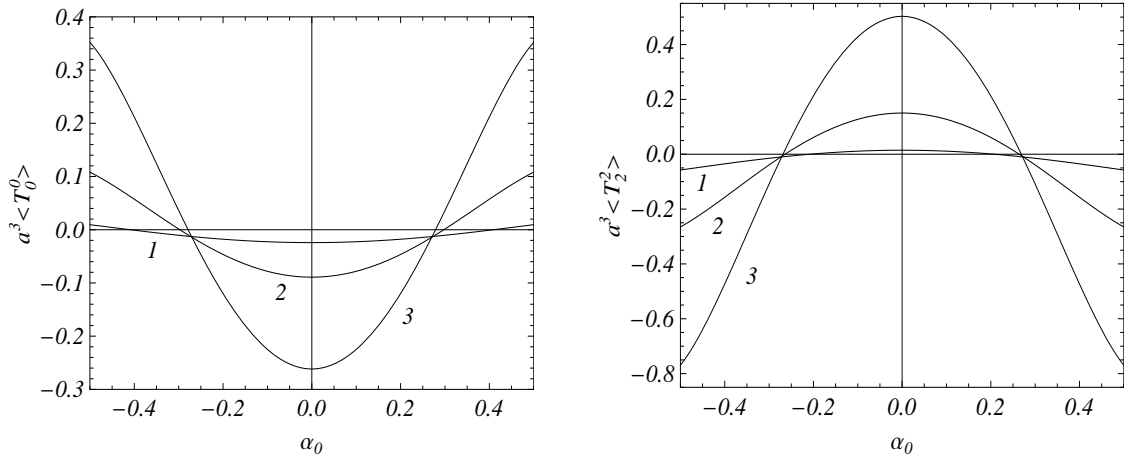


Figure 3: Energy density (left panel) and the azimuthal stress (right panel) for a massless fermionic field as functions of the parameter α_0 for $r/a = 0.5$. The numbers near the curves correspond to the values of the parameter q .

Various special cases of the general formula (3.12) can be considered. In the absence of the magnetic flux one has $\alpha = 0$ and the contributions of the negative and positive values of j to the VEVs coincide. The corresponding formulas are obtained from Eq. (3.12) making the

replacements

$$\sum_j \rightarrow 2 \sum_{j=1/2, 3/2, \dots}, \quad \beta_j \rightarrow qj - 1/2, \quad \beta_j + \epsilon_j \rightarrow qj + 1/2. \quad (3.23)$$

In the case $q = 1$, we obtain the VEVs induced by the magnetic flux and a circular boundary in the Minkowski spacetime. And finally, in the simplest case $\alpha = 0$ and $q = 1$ one has $\langle T_i^k \rangle_{0, \text{ren}} = 0$, and the expressions (3.12) give the VEVs induced by a circular boundary in the Minkowski bulk:

$$\begin{aligned} \langle T_0^0 \rangle_b &= -\frac{a^{-3}}{\pi^2} \sum_{n=0}^{\infty} \int_{\mu}^{\infty} dx \frac{x \sqrt{x^2 - \mu^2}}{U_{n,n+1}^{(I)}(x)} \\ &\quad \times \left\{ [I_n^2(xr/a) - I_{n+1}^2(xr/a)] W_{n,n+1}^{(+)}(x) + (\mu/x) [I_n^2(xr/a) + I_{n+1}^2(xr/a)] \right\}, \\ \langle T_1^1 \rangle_b &= -\frac{a^{-3}}{\pi^2} \sum_{n=0}^{\infty} \int_{\mu}^{\infty} dx \frac{x^3 W_{n,n+1}^{(+)}(x)}{\sqrt{x^2 - \mu^2}} \frac{I'_n(xr/a) I_{n+1}(xr/a) - I'_{n+1}(xr/a) I_n(xr/a)}{U_{n,n+1}^{(I)}(x)}, \\ \langle T_2^2 \rangle_b &= \frac{a^{-2}}{\pi^2 r} \sum_{n=0}^{\infty} (2n+1) \int_{\mu}^{\infty} dx \frac{x^2 W_{n,n+1}^{(+)}(x)}{\sqrt{x^2 - \mu^2}} \frac{I_n(xr/a) I_{n+1}(xr/a)}{U_{n,n+1}^{(I)}(x)}, \end{aligned} \quad (3.24)$$

where the functions $W_{n,n+1}^{(+)}(x)$ and $U_{n,n+1}^{(I)}(x)$ are defined by Eq. (3.9).

4 VEV outside a circular boundary

In this section we consider the region outside a circular boundary with radius a . The corresponding negative-energy mode functions, obeying the boundary condition (2.4), are given by the expression [21]

$$\psi_{\gamma_j}^{(-)}(x) = c_0 e^{-iqj\phi + iEt} \begin{pmatrix} \frac{\gamma \epsilon_j e^{-iq\phi/2}}{E+m} g_{\beta_j, \beta_j + \epsilon_j}(\gamma a, \gamma r) \\ g_{\beta_j, \beta_j}(\gamma a, \gamma r) e^{iq\phi/2} \end{pmatrix}, \quad (4.1)$$

with the function

$$g_{\nu, \rho}(x, y) = \bar{Y}_{\nu}^{(-)}(x) J_{\rho}(y) - \bar{J}_{\nu}^{(-)}(x) Y_{\rho}(y), \quad (4.2)$$

and with $Y_{\nu}(x)$ being the Neumann function. The barred notation is defined by the relation

$$\bar{F}_{\beta_j}^{(-)}(z) = -\epsilon_j z F_{\beta_j + \epsilon_j}(z) - (\sqrt{z^2 + \mu^2} + \mu) F_{\beta_j}(z), \quad (4.3)$$

with $F = J, Y$ and, as before, $\mu = ma$. For the normalization coefficient in Eq. (4.1) one has

$$c_0^2 = \frac{2E\gamma}{\phi_0(E+m)} [\bar{J}_{\beta_j}^{(-)2}(\gamma a) + \bar{Y}_{\beta_j}^{(-)2}(\gamma a)]^{-1}. \quad (4.4)$$

The positive-energy eigenspinors are obtained by making use of the relation $\psi_{\gamma_n}^{(+)} = \sigma_1 \psi_{\gamma_n}^{(-)*}$. Note that in the exterior region the conical singularity is excluded by the boundary and all modes described by eigenspinors (4.1) are regular.

Substituting the mode functions into the mode-sum formula, the VEVs for separate components of the energy-momentum tensor are written in the form

$$\begin{aligned}
\langle T_0^0 \rangle &= -\frac{q}{4\pi} \sum_j \int_0^\infty d\gamma \gamma \frac{(E-m)g_{\beta_j, \beta_j+\epsilon_j}^2(\gamma a, \gamma r) + (E+m)g_{\beta_j, \beta_j}^2(\gamma a, \gamma r)}{\bar{J}_{\beta_j}^{(-)2}(\gamma a) + \bar{Y}_{\beta_j}^{(-)2}(\gamma a)}, \\
\langle T_1^1 \rangle &= \frac{q}{4\pi} \sum_j \int_0^\infty d\gamma \frac{\gamma^3}{E} \frac{g_{\beta_j, \beta_j+\epsilon_j}^2(\gamma a, \gamma r) + g_{\beta_j, \beta_j}^2(\gamma a, \gamma r)}{\bar{J}_{\beta_j}^{(-)2}(\gamma a) + \bar{Y}_{\beta_j}^{(-)2}(\gamma a)} - \langle T_2^2 \rangle, \\
\langle T_2^2 \rangle &= \frac{q}{4\pi r} \sum_j \int_0^\infty d\gamma \frac{(2\beta_j + \epsilon_j)\gamma^2/E}{\bar{J}_{\beta_j}^{(-)2}(\gamma a) + \bar{Y}_{\beta_j}^{(-)2}(\gamma a)} g_{\beta_j, \beta_j}(\gamma a, \gamma r) g_{\beta_j, \beta_j+\epsilon_j}(\gamma a, \gamma r).
\end{aligned} \tag{4.5}$$

As before, we assume the presence of a cutoff function which makes the expression on the right-hand sides of Eq. (4.5) finite. Similar to the interior region, the VEVs outside a circular boundary may be written in the decomposed form (3.11).

In order to find an explicit expression for the boundary-induced part, we note that the boundary-free part is given by Eq. (2.10). For the evaluation of the difference between the total VEV and the boundary-free part, we use the identities

$$\begin{aligned}
\frac{g_{\beta_j, \lambda}^2(x, y)}{\bar{J}_{\beta_j}^{(-)2}(x) + \bar{Y}_{\beta_j}^{(-)2}(x)} &= J_\lambda^2(y) - \frac{1}{2} \sum_{l=1,2} \frac{\bar{J}_{\beta_j}^{(-)}(x)}{\bar{H}_{\beta_j}^{(-,l)}(x)} H_\lambda^{(l)2}(y), \\
\frac{g_{\beta_j, \lambda}(x, y) g_{\beta_j, \lambda+\epsilon_j}(x, y)}{\bar{J}_{\beta_j}^{(-)2}(x) + \bar{Y}_{\beta_j}^{(-)2}(x)} &= J_\lambda(y) J_{\lambda+\epsilon_j}(y) - \frac{1}{2} \sum_{l=1,2} \frac{\bar{J}_{\beta_j}^{(-)}(x)}{\bar{H}_{\beta_j}^{(-,l)}(x)} H_\lambda^{(l)}(y) H_{\lambda+\epsilon_j}^{(l)}(y),
\end{aligned} \tag{4.6}$$

with $\lambda = \beta_j, \beta_j + \epsilon_j$, and with $H_\nu^{(l)}(x)$ being the Hankel function.

In this way, for the boundary induced parts we find the expressions

$$\begin{aligned}
\langle T_0^0 \rangle_b &= \frac{q}{8\pi} \sum_j \sum_{l=1,2} \int_0^\infty d\gamma \gamma \frac{\bar{J}_{\beta_j}^{(-)}(\gamma a)}{\bar{H}_{\beta_j}^{(-,l)}(\gamma a x)} \left[(E-m) H_{\beta_j+\epsilon_j}^{(l)2}(\gamma r) + (E+m) H_{\beta_j}^{(l)2}(\gamma r) \right], \\
\langle T_1^1 \rangle_b &= -\frac{q}{8\pi} \sum_j \sum_{l=1,2} \int_0^\infty d\gamma \frac{\gamma^3}{E} \frac{\bar{J}_{\beta_j}^{(-)}(\gamma a)}{\bar{H}_{\beta_j}^{(-,l)}(\gamma a)} \left[H_{\beta_j+\epsilon_j}^{(l)2}(\gamma r) + H_{\beta_j}^{(l)2}(\gamma r) \right] - \langle T_2^2 \rangle_b, \\
\langle T_2^2 \rangle_b &= -\frac{q}{8\pi r} \sum_j (2\beta_j + \epsilon_j) \sum_{l=1,2} \int_0^\infty d\gamma \frac{\gamma^2}{E} \frac{\bar{J}_{\beta_j}^{(-)}(\gamma a)}{\bar{H}_{\beta_j}^{(-,l)}(\gamma a)} H_{\beta_j}^{(l)}(\gamma r) H_{\beta_j+\epsilon_j}^{(l)}(\gamma r).
\end{aligned} \tag{4.7}$$

In the complex plane γ , the integrand of the term with $l = 1$ ($l = 2$) decays exponentially in the limit $\text{Im}(\gamma) \rightarrow \infty$ [$\text{Im}(\gamma) \rightarrow -\infty$] for $r > a$. By using these properties, we rotate the integration contour in the complex plane γ by the angle $\pi/2$ for the term with $l = 1$ and by the angle $-\pi/2$ for the term with $l = 2$. The integrals over the segments $(0, im)$ and $(0, -im)$ of the imaginary axis cancel each other. Introducing the modified Bessel functions, the boundary-induced parts

are presented in the form

$$\begin{aligned}
\langle T_0^0 \rangle_b &= -\frac{q}{2\pi^2} \sum_j \int_m^\infty dx x \frac{\sqrt{x^2 - m^2}}{U_{\beta_j, \beta_j + \epsilon_j}^{(K)}(ax)} \left\{ m[K_{\beta_j + \epsilon_j}^2(xr) + K_{\beta_j}^2(xr)] \right. \\
&\quad \left. + x[K_{\beta_j}^2(xr) - K_{\beta_j + \epsilon_j}^2(xr)] W_{\beta_j, \beta_j + \epsilon_j}^{(-)}(ax) \right\}, \\
\langle T_1^1 \rangle_b &= -\frac{q}{2\pi^2} \sum_j \int_m^\infty dx x^3 \frac{K_{\beta_j + \epsilon_j}^2(xr) - K_{\beta_j}^2(xr)}{\sqrt{x^2 - m^2}} \frac{W_{\beta_j, \beta_j + \epsilon_j}^{(-)}(ax)}{U_{\beta_j, \beta_j + \epsilon_j}^{(K)}(ax)} - \langle T_2^2 \rangle_b, \\
\langle T_2^2 \rangle_b &= -\frac{q}{2\pi^2 r} \sum_j (2\beta_j \epsilon_j + 1) \int_m^\infty dx x^2 \frac{K_{\beta_j}(xr) K_{\beta_j + \epsilon_j}(xr)}{\sqrt{x^2 - m^2}} \frac{W_{\beta_j, \beta_j + \epsilon_j}^{(-)}(ax)}{U_{\beta_j, \beta_j + \epsilon_j}^{(K)}(ax)}.
\end{aligned} \tag{4.8}$$

Here we have introduced the notation

$$U_{\nu, \sigma}^{(K)}(x) = x[K_\nu^2(x) + K_\sigma^2(x)] + 2\mu K_\nu(x) K_\sigma(x), \tag{4.9}$$

and the notation $W_{\beta_j, \beta_j + \epsilon_j}^{(-)}(x)$ is defined by Eq. (3.10). By taking into account that under the change $\alpha \rightarrow -\alpha$, $j \rightarrow -j$, one has $\beta_j \rightarrow \beta_j + \epsilon_j$, $\beta_j + \epsilon_j \rightarrow \beta_j$, we conclude that $W_{\beta_j, \beta_j + \epsilon_j}^{(-)}(x)$ and $U_{\beta_j, \beta_j + \epsilon_j}^{(K)}(x)$ are odd and even functions under this change. Now, from Eq. (4.8) it follows that the boundary-induced parts are even functions of α . They are periodic with the period equal to 1.

For a massless field the expressions for the boundary-induced parts in the VEVs simplify to

$$\begin{aligned}
\langle T_0^0 \rangle_b &= -\frac{q}{2\pi^2 a^3} \sum_j \int_0^\infty dx x^2 V_{\beta_j, \beta_j + \epsilon_j}^{(K)}(x) [K_{\beta_j}^2(xr/a) - K_{\beta_j + \epsilon_j}^2(xr/a)], \\
\langle T_2^2 \rangle_b &= -\frac{q}{2\pi^2 a^2 r} \sum_j (2\beta_j \epsilon_j + 1) \int_0^\infty dx x V_{\beta_j, \beta_j + \epsilon_j}^{(K)}(x) K_{\beta_j}(xr/a) K_{\beta_j + \epsilon_j}(xr/a),
\end{aligned} \tag{4.10}$$

with the notation

$$V_{\nu, \sigma}^{(K)}(x) = \frac{I_\nu(x) K_\nu(x) - I_\sigma(x) K_\sigma(x)}{K_\nu^2(x) + K_\sigma^2(x)}. \tag{4.11}$$

For the radial stress one has $\langle T_1^1 \rangle_b = -\langle T_0^0 \rangle_b - \langle T_2^2 \rangle_b$. In particular, for the circle in the Minkowski bulk the corresponding formulas are obtained from Eq. (3.24) by the interchange $I \rightleftharpoons K$, replacing $W_{n, n+1}^{(+)}(x) \rightarrow W_{n, n+1}^{(-)}(x)$.

Now we turn to the investigation of the VEVs in the asymptotic regions for the parameters. First we consider the limit $a \rightarrow 0$ for a fixed value of r . By using the asymptotic formulas for the modified Bessel functions for small arguments, we can see that the dominant contribution comes from the term with $j = 1/2$ for $-1/2 < \alpha_0 < 0$ and from the term $j = 1/2$ for $0 < \alpha_0 < 1/2$. For a massive field to the leading order we get (no summation over i)

$$\langle T_i^i \rangle_b \approx -\frac{qm}{\pi^2 r^2} \frac{(a/2r)^{2q_\alpha}}{\Gamma^2(q_\alpha + 1/2)} \int_{mr}^\infty dx \frac{x^{2q_\alpha+2} Z_i(x)}{\sqrt{x^2 - m^2 r^2}}, \tag{4.12}$$

with the notations

$$\begin{aligned}
Z_0(x) &= (1 - m^2 r^2 / x^2) K_{q_\alpha - 1/2}^2(x), \\
Z_1(x) &= K_{q_\alpha + 1/2}^2(x) - K_{q_\alpha - 1/2}^2(x) - Z_2(x), \\
Z_2(x) &= \frac{2q_\alpha}{x} K_{q_\alpha - 1/2}(x) K_{q_\alpha + 1/2}(x),
\end{aligned} \tag{4.13}$$

and q_α is defined by Eq. (3.17). We see that the boundary-induced part vanishes in the limit $a \rightarrow 0$ for $|\alpha_0| < 1/2$. For a massless field the leading term in the corresponding asymptotic expansion for the azimuthal stress is given by the expression

$$\langle T_2^2 \rangle_b = -\frac{q}{4\pi r^3} \frac{(a/2r)^{2q_\alpha+1}}{q_\alpha^2 - 1/4} \frac{q_\alpha \Gamma(q_\alpha + 1) \Gamma(2q_\alpha + 3/2)}{\Gamma(q_\alpha + 3/2) \Gamma^2(q_\alpha + 1/2)}, \quad (4.14)$$

For the energy density and the radial stress one has the relations

$$\langle T_0^0 \rangle_b \approx -\frac{q_\alpha + 1}{q_\alpha + 3/2} \langle T_2^2 \rangle_b, \quad \langle T_1^1 \rangle_b \approx \frac{-\langle T_2^2 \rangle_b}{2(q_\alpha + 3/2)}. \quad (4.15)$$

For a massless field the boundary-induced VEVs are suppressed with an additional factor a/r with respect to the case of a massive field.

Let us consider the limit of large distances from the circle. For a massive field, assuming $mr \gg 1$, we see that the dominant contribution to the integrals in Eq. (4.8) comes from the lower limit of the integration. In the leading order we obtain:

$$\begin{aligned} \langle T_0^0 \rangle_b &\approx -\frac{qm^3 e^{-2mr}}{8\sqrt{\pi}(mr)^{5/2}} \sum_j \frac{1}{U_{\beta_j, \beta_j + \epsilon_j}^{(K)}(am)}, \\ \langle T_2^2 \rangle_b &\approx -\frac{qm^3 e^{-2mr}}{8\sqrt{\pi}(mr)^{5/2}} \sum_j \frac{(2\beta_j \epsilon_j + 1) W_{\beta_j, \beta_j + \epsilon_j}^{(-)}(am)}{U_{\beta_j, \beta_j + \epsilon_j}^{(K)}(am)}, \end{aligned} \quad (4.16)$$

and for the radial stress one has $\langle T_1^1 \rangle_b = -\langle T_2^2 \rangle_b / (2mr)$. As we could expect, in this limit the VEVs are exponentially suppressed. The radial stress contains an additional suppression factor $(mr)^{-1}$. For a massless field the leading terms for $r \gg a$ are given by Eqs. (4.14) and (4.15).

It remains to consider the behavior of the VEVs near the boundary. In this region the dominant contribution comes from large values of $|j|$ and, in the way similar to that for the interior region, we find

$$\begin{aligned} \langle T_0^0 \rangle_b &\approx \frac{1/8 - \mu}{16\pi a(r-a)^2}, \\ \langle T_2^2 \rangle_b &\approx \frac{a}{a-r} \langle T_1^1 \rangle_b \approx -\frac{1/8 + \mu}{16\pi a(r-a)^2}. \end{aligned} \quad (4.17)$$

Comparing with Eq. (3.21), we see that for a massless field the energy density and the azimuthal stress have opposite signs for the exterior and interior regions, whereas the radial stress has the same sign.

The boundary-induced parts in the exterior region are displayed in Fig. 4 as functions of the radial coordinate. The full and dashed curves are for the energy density and the azimuthal stress, respectively, and the numbers near the curves correspond to the values of q . The left and right panels are plotted for $\alpha_0 = 0$ and $\alpha_0 = 0.4$, respectively.

In Fig. 5 we plot the VEVs of the vacuum energy density and the azimuthal stress for a massless field versus α_0 for $r/a = 2$. The numbers near the curves correspond to the values of the parameter q .

The results given above can be applied to graphitic cones within the framework of long-wavelength Dirac-like model for electronic states in graphene (for a review see [34]). Graphene made structures have attracted much attention recently due to the experimental observation of a number of novel electronic properties. The electronic band structure of graphene close to the Dirac points shows a conical dispersion $E(\mathbf{k}) = v_F |\mathbf{k}|$, where \mathbf{k} is the momentum measured

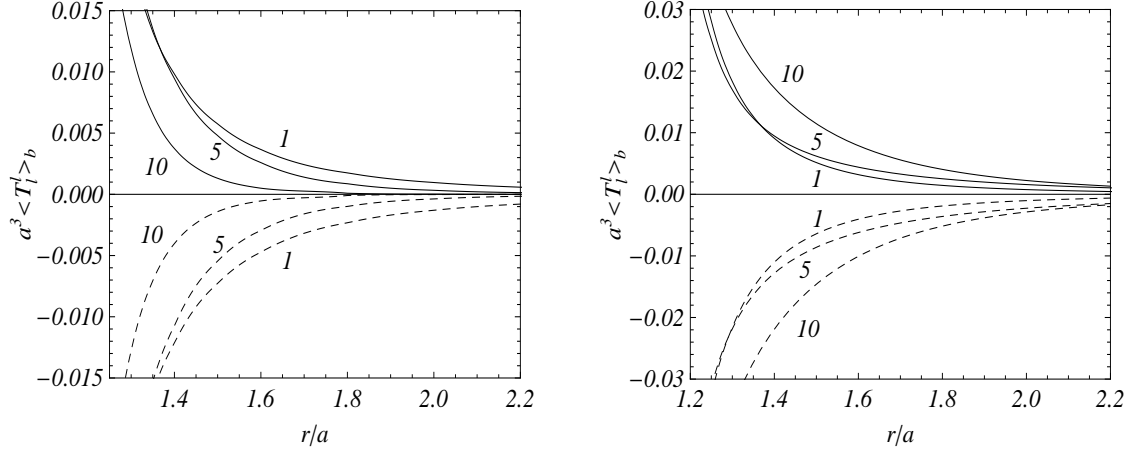


Figure 4: The same as in Fig. 2 for the region outside a circular boundary.

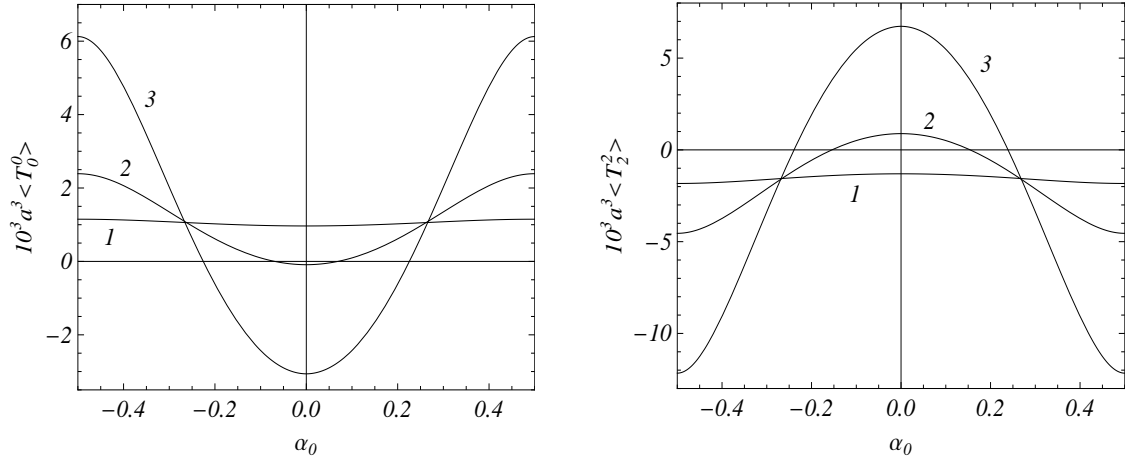


Figure 5: The same as in Fig. 3 for the region outside a circular boundary with $r/a = 2$.

relatively to the Dirac points and $v_F \approx 10^8$ cm/s represents the Fermi velocity which plays the role of a speed of light. The low-energy excitations can be described by a pair of two-component spinors, ψ_J , $J = 1, 2$, corresponding to the two inequivalent Fermi points of the Brillouin zone. The components of the spinor ψ_J correspond to two triangular sublattices of the honeycomb lattice of graphene. For a flat graphene sheet, in the absence of interactions that mix the inequivalent Fermi points, the electronic states attached to these points will be independent. The Dirac equation for the corresponding spinors has the form

$$(iv_F^{-1}\gamma^0 D_0 + i\gamma^l D_l - m)\psi_J = 0, \quad (4.18)$$

where $l = 1, 2$, and $D_\mu = \nabla_\mu + ieA_\mu$ with $e = -|e|$ for electrons. The mass (gap) term in (4.18) is essential in many physical application. This gap can be generated by a number of mechanisms. In particular, they include the breaking of symmetry between two sublattices by introducing a staggered onsite energy [42] and the deformations of bonds in the graphene lattice [43]. Another approach is to attach a graphene monolayer to a substrate the interaction with which breaks the sublattice symmetry [44]. Graphitic cones are obtained from the graphene sheet if one or more sectors with the angle $\pi/6$ are removed. The opening angle of the cone is related to the number of sectors removed, N_c , by the formula $2\pi(1 - N_c/6)$, with $N_c = 1, 2, \dots, 5$ (for the electronic properties of graphitic cones see, e.g., [45] and references therein). All these angles have been observed in experiments [46]. For even values of N_c the periodicity conditions do not mix the spinors ψ_1 and ψ_2 . The corresponding expressions for the Casimir densities for finite radius graphitic nanocones are obtained from the formulas given above with additional factor 2 which takes into account the presence of two inequivalent Fermi points. In standard units, the factor $\hbar v_F$ appears as well. For odd values of N_c the periodicity condition mixes the spinors corresponding to inequivalent Fermi points. In this case it is convenient to combine two spinors ψ_1 and ψ_2 in a single bispinor. The evaluation for the corresponding Casimir densities can be done in a way similar to that described before.

5 Half-integer values of the parameter α

In this section we consider the VEV of the energy-momentum tensor for half-integer values of the parameter α . In this case the mode with $j = -\alpha$ must be considered separately.

5.1 Boundary-free part

For half-integer values of α , in the boundary-free geometry the eigenspinors with $j \neq -\alpha$ are still given by Eq. (2.6). For the mode function corresponding to the special mode with $j = -\alpha$ one has [21]

$$\psi_{(0)\gamma, -\alpha}^{(-)}(x) = \left(\frac{E+m}{\pi\phi_0 r E}\right)^{1/2} e^{iq\alpha\phi + iEt} \begin{pmatrix} \frac{\gamma e^{-iq\phi/2}}{E+m} \sin(\gamma r - \gamma_0) \\ e^{iq\phi/2} \cos(\gamma r - \gamma_0) \end{pmatrix}, \quad (5.1)$$

where, as before, $E = \sqrt{\gamma^2 + m^2}$ and we have defined

$$\gamma_0 = \arccos[\sqrt{(E-m)/2E}]. \quad (5.2)$$

As it has been noted above, for half-integer values of α the mode with $j = -\alpha$ corresponds to the irregular mode. The contribution of the modes with $j \neq -\alpha$ to the VEV of the energy-momentum tensor remains the same as before. Special consideration is needed for the mode

with $j = -\alpha$ only. For the contribution of this mode to the VEVs of the energy density and the radial stress we have the expressions

$$\begin{aligned}\langle T_0^0 \rangle_0^{(j=-\alpha)} &= -\frac{q}{2\pi^2 r} \int_0^\infty d\gamma [E + m \cos(2\gamma r - 2\gamma_0)], \\ \langle T_1^1 \rangle_0^{(j=-\alpha)} &= \frac{q}{2\pi^2 r} \int_0^\infty d\gamma (E - m^2/E),\end{aligned}\quad (5.3)$$

and the contribution to the azimuthal stress vanishes. As we have done in Sect. 2, for the regularization of the expressions (5.3) we introduce the cutoff function $e^{-s\gamma^2}$. After the integration we find the following expressions

$$\begin{aligned}\langle T_0^0 \rangle_{0,\text{reg}}^{(j=-\alpha)} &= -\frac{qm^2}{8\pi^2 r} \left\{ e^{sm^2/2} [K_0(sm^2/2) + K_1(sm^2/2)] \right. \\ &\quad \left. + 4K_1(2mr) - 4K_0(2mr) + o(s) \right\}, \\ \langle T_1^1 \rangle_{0,\text{reg}}^{(j=-\alpha)} &= -\frac{qm^2}{8\pi^2 r} e^{sm^2/2} [K_0(sm^2/2) - K_1(sm^2/2)] + o(s).\end{aligned}\quad (5.4)$$

In order to obtain the total VEV we should add the regularized part corresponding to the modes with $j \neq -\alpha$. For half-integer values of α , for the series in the contribution of these modes one has

$$\sum_{j \neq -\alpha} I_{\beta_j}(x) = \sum_{j \neq -\alpha} I_{\beta_j + \epsilon_j}(x) = \sum_{n=1}^{\infty} [I_{qn-1/2}(x) + I_{qn+1/2}(x)]. \quad (5.5)$$

As a result, for this part in the regularized VEV of the radial stress one finds

$$\langle T_1^1 \rangle_{0,\text{reg}}^{(j \neq -\alpha)} = \frac{qe^{m^2 s}}{(2\pi)^{3/2}} \int_0^{1/(2s)} dy \frac{y^{1/2} e^{-m^2/(2y) - r^2 y}}{\sqrt{1-2ys}} \sum_{n=1}^{\infty} [I_{qn-1/2}(r^2 y) + I_{qn+1/2}(r^2 y)]. \quad (5.6)$$

The corresponding parts in the energy density and the azimuthal stress are given by the relations

$$\begin{aligned}\langle T_0^0 \rangle_{0,\text{reg}}^{(j \neq -\alpha)} &= -(2 + r\partial_r) \langle T_1^1 \rangle_{0,\text{reg}}^{(j \neq -\alpha)} + m \langle \bar{\psi}\psi \rangle_{0,\text{reg}}^{(j \neq -\alpha)}, \\ \langle T_2^2 \rangle_{0,\text{reg}}^{(j \neq -\alpha)} &= (1 + r\partial_r) \langle T_1^1 \rangle_{0,\text{reg}}^{(j \neq -\alpha)},\end{aligned}\quad (5.7)$$

where

$$\begin{aligned}\langle \bar{\psi}\psi \rangle_{0,\text{reg}}^{(j \neq -\alpha)} &= -\frac{qme^{m^2 s}}{(2\pi)^{3/2}} \int_0^{r^2/2s} dy \frac{y^{-1/2} e^{-m^2/2y - r^2 y}}{\sqrt{1-2ys}} \\ &\quad \times \sum_{n=1}^{\infty} [I_{qn-1/2}(r^2 y) + I_{qn+1/2}(r^2 y)].\end{aligned}\quad (5.8)$$

The fermionic condensate has been considered in Ref. [22], and here we need to consider the radial stress only. After the summation over n by using the formula given in Sect. 2, we find the following representation

$$\begin{aligned}\langle T_1^1 \rangle_{0,\text{reg}}^{(j \neq -\alpha)} &= \frac{e^{m^2 s}}{(2\pi)^{3/2}} \int_0^{1/2s} dy \frac{y^{1/2} e^{-m^2/2y}}{\sqrt{1-2ys}} + \frac{qm^2}{8\pi^2 r} e^{sm^2/2} [K_0(sm^2/2) - K_1(sm^2/2)] \\ &\quad + \frac{m^3}{\pi} e^{m^2 s} \left[\sum_{l=1}^p \cos(\pi l/q) F_1^{(s)}(2mr s_l) \right. \\ &\quad \left. + \frac{q}{\pi} \int_0^\infty dy \frac{\sinh(y) \sinh(2qy)}{\cosh(2qy) - \cos(q\pi)} F_1^{(s)}(2mr \cosh y) \right],\end{aligned}\quad (5.9)$$

where $2p \leq q < 2p+2$. Note that the second term in the right-hand side of this formula does not contribute to the azimuthal stress. Comparing with (5.4), we see that in the total regularized VEV this part is cancelled by the part coming from the irregular mode. As a result, for the total regularized VEV one finds the expression

$$\begin{aligned} \langle T_1^1 \rangle_{0,\text{reg}} &= \frac{e^{m^2 s}}{(2\pi)^{3/2}} \int_0^{1/2s} dy \frac{y^{1/2} e^{-m^2/2y}}{\sqrt{1-2ys}} + \frac{m^3}{\pi} e^{m^2 s} \left[\sum_{l=1}^p \cos(\pi l/q) F_1^{(s)}(2mr s_l) \right. \\ &\quad \left. + \frac{q}{\pi} \int_0^\infty dy \frac{\sinh(y) \sinh(2qy)}{\cosh(2qy) - \cos(q\pi)} F_1^{(s)}(2mr \cosh y) \right] + o(s). \end{aligned} \quad (5.10)$$

The first term in the right-hand side of this expression corresponds to the contribution coming from the Minkowski spacetime part. It is subtracted in the renormalization procedure and for the renormalized VEV of the radial stress in a boundary-free conical space one finds

$$\begin{aligned} \langle T_1^1 \rangle_{0,\text{ren}} &= \frac{m^3}{\pi} \left[\sum_{l=1}^p \cos(\pi l/q) F_1^{(s)}(2mr s_l) \right. \\ &\quad \left. + \frac{q}{\pi} \int_0^\infty dy \frac{\sinh(y) \sinh(2qy)}{\cosh(2qy) - \cos(q\pi)} F_1^{(s)}(2mr \cosh y) \right]. \end{aligned} \quad (5.11)$$

Combining the results (5.4), (5.7), and (5.10), for the renormalized VEVs of the energy density and the azimuthal stress we obtain the expressions

$$\begin{aligned} \langle T_0^0 \rangle_{0,\text{ren}} &= \langle T_1^1 \rangle_{0,\text{ren}} - \frac{qm^2}{2\pi^2 r} [K_1(2mr) - K_0(2mr)], \\ \langle T_2^2 \rangle_{0,\text{ren}} &= \frac{m^3}{\pi} \left\{ \sum_{l=1}^p \cos(\pi l/q) F_2^{(s)}(2mr s_l) \right. \\ &\quad \left. + \frac{q}{\pi} \int_0^\infty dy \frac{\sinh(y) \sinh(2qy)}{\cosh(2qy) - \cos(q\pi)} F_2^{(s)}(2mr \cosh(y)) \right\}, \end{aligned} \quad (5.12)$$

where the function $F_2^{(s)}(x)$ is defined by Eq. (2.32). As we see, for half-integer values of α , when an irregular mode is present, the energy density and the radial stress differ. In the case of a massless field the radial stress is expressed as

$$\langle T_1^1 \rangle_{0,\text{ren}} = \frac{1}{8\pi r^3} \left[\sum_{l=1}^p \frac{\cos(\pi l/q)}{s_l^3} + \frac{q}{\pi} \int_0^\infty dy \frac{\sinh(y) \sinh(2qy)}{\cosh(2qy) - \cos(q\pi)} \frac{1}{\cosh^3 y} \right], \quad (5.13)$$

and for the energy density and the azimuthal stress one has:

$$\langle T_0^0 \rangle_{0,\text{ren}} = \langle T_1^1 \rangle_{0,\text{ren}} = -\langle T_2^2 \rangle_{0,\text{ren}}/2. \quad (5.14)$$

Of course, in this case the renormalized VEV is traceless.

5.2 Region inside a circular boundary

Now we consider the region inside a circle with radius a . The contribution of the modes with $j \neq -\alpha$ is given by Eq. (3.12) where now the summation goes over $j \neq -\alpha$. For the evaluation of the contribution coming from the mode with $j = -\alpha$, we note that the negative-energy eigenspinor for this mode has the form [21]

$$\psi_{\gamma, -\alpha}^{(-)}(x) = \frac{b_0}{\sqrt{r}} e^{iq\alpha\phi + iEt} \begin{pmatrix} \frac{\gamma e^{-iq\phi/2}}{E+m} \sin(\gamma r - \gamma_0) \\ e^{iq\phi/2} \cos(\gamma r - \gamma_0) \end{pmatrix}, \quad (5.15)$$

where γ_0 is defined by Eq. (5.2). From boundary condition (2.4) it follows that the eigenvalues of γ are solutions of the equation

$$m \sin(\gamma a) + \gamma \cos(\gamma a) = 0. \quad (5.16)$$

We denote the positive roots of this equation by $\gamma_l = \gamma a$, $l = 1, 2, \dots$. From the normalization condition, for the coefficient in Eq. (5.15) one has

$$b_0^2 = \frac{E + m}{aE\phi_0} [1 - \sin(2\gamma a)/(2\gamma a)]^{-1}. \quad (5.17)$$

Using Eq. (5.15), for the contributions of the mode under consideration to the energy density and the radial stress we find:

$$\begin{aligned} \langle T_0^0 \rangle_{j=-\alpha} &= -\frac{q}{2\pi a^2 r} \sum_{l=1}^{\infty} \frac{\gamma_l^2 + \mu^2 + \mu [\gamma_l \sin(2\gamma_l r/a) - \mu \cos(2\gamma_l r/a)]}{\sqrt{\gamma_l^2 + \mu^2} [1 - \sin(2\gamma_l)/(2\gamma_l)]}, \\ \langle T_1^1 \rangle_{j=-\alpha} &= \frac{q}{2\pi a^2 r} \sum_{l=1}^{\infty} \frac{\gamma_l^2 / \sqrt{\gamma_l^2 + \mu^2}}{1 - \sin(2\gamma_l)/(2\gamma_l)}, \end{aligned} \quad (5.18)$$

where $\mu = ma$ and the presence of a cutoff function is assumed. The contribution to the azimuthal stress vanishes: $\langle T_2^2 \rangle_{j=-\alpha} = 0$. For the summation of the series in Eq. (5.18), we use the Abel-Plana-type formula [40, 47]

$$\sum_{l=1}^{\infty} \frac{\pi f(\gamma_l)}{1 - \sin(2\gamma_l)/(2\gamma_l)} = -\frac{\pi f(0)/2}{1/\mu + 1} + \int_0^{\infty} dz f(z) - i \int_0^{\infty} dz \frac{f(iz) - f(-iz)}{\frac{z+\mu}{z-\mu} e^{2z} + 1}. \quad (5.19)$$

For the functions $f(z)$ corresponding to Eq. (5.18) one has $f(0) = 0$. The second term on the right-hand side of Eq. (5.19) gives the part corresponding to the boundary-free geometry. As a result, the VEVs are presented in the decomposed form (no summation)

$$\langle T_i^i \rangle_{j=-\alpha} = \langle T_i^i \rangle_0^{(j=-\alpha)} + \langle T_i^i \rangle_{b,j=-\alpha}, \quad (5.20)$$

where the boundary-induced parts are given by the expressions

$$\begin{aligned} \langle T_0^0 \rangle_{b,j=-\alpha} &= -\frac{q}{\pi^2 r} \int_m^{\infty} dx \frac{x^2 - m^2 + m [x \sinh(2xr) + m \cosh(2xr)]}{\sqrt{x^2 - m^2} \left(\frac{x+m}{x-m} e^{2ax} + 1 \right)}, \\ \langle T_1^1 \rangle_{b,j=-\alpha} &= \frac{q}{\pi^2 r} \int_m^{\infty} dx \frac{x^2 / \sqrt{x^2 - m^2}}{\frac{x+m}{x-m} e^{2ax} + 1}. \end{aligned} \quad (5.21)$$

Note that

$$\langle T_0^0 \rangle_{b,j=-\alpha} = -\langle T_1^1 \rangle_{b,j=-\alpha} + m \langle \bar{\psi} \psi \rangle_{b,j=-\alpha}, \quad (5.22)$$

where

$$\langle \bar{\psi} \psi \rangle_{b,j=-\alpha} = \frac{q}{\pi^2 r} \int_m^{\infty} dx \frac{m - x \sinh(2xr) - m \cosh(2xr)}{\sqrt{x^2 - m^2} \left(\frac{x+m}{x-m} e^{2ax} + 1 \right)}, \quad (5.23)$$

is the corresponding part in the fermionic condensate. The contribution of the modes $j \neq -\alpha$ remains the same and is obtained from the corresponding expressions given above for non-half-integer values of α by the direct substitution $\alpha = 1/2$.

Expression (5.21) for the boundary induced part is simplified for a massless field:

$$\langle T_0^0 \rangle_{b,j=-\alpha} = -\langle T_1^1 \rangle_{b,j=-\alpha} = \frac{q}{48a^2r}. \quad (5.24)$$

Unlike to the fermionic condensate, the boundary induced VEVs diverge at the circle center. Note that for a massless field these VEVs are finite on the boundary. Adding the part corresponding to the regular modes, for the total VEVs in the massless case we get

$$\begin{aligned} \langle T_0^0 \rangle_b &= \frac{q}{48a^2r} - \frac{q}{\pi^2 a^3} \sum_{n=1}^{\infty} \int_0^{\infty} dx x^2 V_{qn-1/2, qn+1/2}^{(I)}(x) [I_{qn-1/2}^2(xr/a) - I_{qn+1/2}^2(xr/a)], \\ \langle T_2^2 \rangle_b &= \frac{2q^2}{\pi^2 a^2 r} \sum_{n=1}^{\infty} n \int_0^{\infty} dx x V_{qn-1/2, qn+1/2}^{(I)}(x) I_{qn-1/2}(xr/a) I_{qn+1/2}(xr/a), \end{aligned} \quad (5.25)$$

and for the radial stress one has $\langle T_1^1 \rangle_b = -\langle T_0^0 \rangle_b - \langle T_2^2 \rangle_b$. The boundary-free parts in this case are given by Eqs. (5.13) and (5.14).

In the region outside a circular boundary there are no irregular modes and the VEV of the energy-momentum tensor is a continuous function of the parameter α at half-integer values. The corresponding expression is obtained taking the limit $\alpha_0 \rightarrow 1/2$ in the expressions for the VEVs given above for $\alpha_0 \neq 1/2$.

6 Conclusion

We have investigated the VEV of the energy-momentum tensor for a massive fermionic field in a (2+1)-dimensional conical spacetime with a circular boundary on which the field obeys MIT bag boundary condition. In addition, we have assumed the presence of magnetic flux located at the cone apex. A special case of boundary conditions at the apex is considered when the MIT bag boundary condition is imposed at a finite radius, which is then taken to zero. In the presence of a circular boundary, the VEV of the energy-momentum tensor is decomposed into the boundary-free and boundary-induced parts.

First we consider the geometry of a conical space without boundaries. The corresponding VEV is evaluated by making use of mode sum formula (2.5) with the eigenspinors given by Eq. (2.6). For the regularization of the mode sums we have introduced an exponential cutoff function. The application of the formula (2.26) allowed us to explicitly extract from the VEVs the parts corresponding to the Minkowski spacetime in the absence of the magnetic flux. The renormalization is reduced to the subtraction of this part. The renormalized VEVs in the boundary-free geometry are given by Eq. (2.30). These VEVs are even and periodic functions of the parameter α , related to the magnetic flux by Eq. (2.8). The corresponding radial stress is equal to the energy density. For a massless field the renormalized VEV of the energy density is expressed as Eq. (2.34), and the azimuthal stress is obtained from the zero-trace condition. In the special case of integer q and for the parameter α given by Eq. (2.37), the general formula is reduced to Eq. (2.38). In this case, the renormalized VEV vanishes in a conical space with $q = 2$. Various other special cases are considered. In particular, for the magnetic flux in background of Minkowski spacetime one has the expressions (2.42). In this case the corresponding energy density is positive.

The effects induced by a circular boundary, concentric with the cone apex, are considered in Sect. 3. In the interior region the eigenvalues for γ are quantized by the boundary condition and they are solutions of Eq. (3.2). The mode sums for the separate components of the energy-momentum tensor are given by Eq. (3.5) and contain the summation over these eigenvalues. The

application of the Abel-Plana-type summation formula allows us to extract from the VEVs the parts corresponding to the boundary-free geometry and to present the boundary-induced parts in terms of rapidly convergent integrals suitable for numerical evaluation. The corresponding expressions are given by Eq. (3.12). The boundary-induced parts are even and periodic functions of the parameter α with the period equal to 1. Note that for the boundary-induced part the energy density is not equal to the radial stress. For a massless field the general formulas are reduced to Eq. (3.14) for the energy density and the azimuthal stress. The expression for the radial stress is obtained by using the tracelessness of the energy-momentum tensor. At the cone apex the boundary induced VEVs vanish for $|\alpha_0| < (1 - 1/q)/2$ and they diverge when $|\alpha_0| > (1 - 1/q)/2$. In particular, the VEVs are divergent for a magnetic flux in background of Minkowski spacetime. Near the boundary, the boundary-induced parts in the VEVs dominate. The leading terms in the asymptotic expansions over the distance from the boundary are given by Eq. (3.21). The energy density is negative near the boundary, whereas the stresses may change the sign with dependence of the field mass.

The vacuum energy-momentum tensor in the region outside a circular boundary is considered in Sect. 4. After the subtraction of the boundary-free parts and by making use of complex rotation, we present the boundary-induced parts in the form (4.8) and (4.10), for massive and massless fields respectively. As in the interior region, they are even and periodic functions of the parameter α with the period equal to 1. For small values of the circle radius when the radial distance is fixed, the boundary-induced VEVs for a massive field behave as $(a/r)^{2q_\alpha}$ with q_α defined by Eq. (3.17). For a massless field the leading terms vanish and the behavior of the VEVs is like $(a/r)^{2q_\alpha+1}$. At large distances from the circle, the boundary induced VEVs are exponentially suppressed for a massive field and they decay as power-law for a massless field. For points near the boundary the VEVs diverge. The leading terms in the expansions over the distance from the boundary are given by Eq. (4.17).

In the case of half-integer values of the parameter α , a special consideration is needed for the mode with $j = -\alpha$. In the boundary-free geometry the corresponding mode functions are given by Eq. (5.1). The contribution of the special mode to the azimuthal stress vanishes. The renormalized expressions are given by Eq. (5.11) for the radial stress and by Eq. (5.12) for the energy density and azimuthal stress. Note that, in the case under consideration the radial stress, in general, is not equal to the energy density. This equality takes place for a massless field only (see Eqs. (5.13) and (5.14)). In the presence of circular boundary and for half-integer values of α , the mode function for the special mode $j = -\alpha$ is given by Eq. (5.15). The corresponding eigenvalues for γ are roots of Eq. (5.16). Similar to the boundary-free case, the contribution of the special mode to the VEV of the azimuthal stress vanishes. The expressions for the energy density and radial stress, Eq. (5.18), are given in terms of series over the eigenvalues of γ . For the summation of these series we use the summation formula (5.19). This allows us to separate the boundary-free part. The boundary-induced parts for the contributions of the special mode are given by Eqs. (5.21) and (5.24) for massive and massless fields, respectively. The total VEVs are obtained adding the parts coming from the modes with $j \neq -\alpha$. The latter are obtained from the formulas given before, putting directly half-integer values for α . In particular, for a massless field we have the expressions (5.25).

From the point of view of the physics in the region outside the conical defect core, the geometry considered in the present paper can be viewed as a simplified model for the non-trivial core. This model presents a framework in which the influence of the finite core effects on physical processes in the vicinity of the conical defect can be investigated. The corresponding results may shed light upon features of finite core effects in more realistic models, including those used for defects in crystals and superfluid helium. In addition, the problem considered here is of interest as an example with combined topological and boundary-induced quantum effects, in which the

physical characteristics can be found in closed analytic form.

Nanocones of carbon appear as a natural environment for applications of the calculations presented in this paper. Like graphene, carbon nanocones, have long-wavelength free electrons which are described effectively as Dirac fermions. Localized defects like the apex and the boundary of the cone act as scatterers producing standing-wave patterns in the electron density. The interaction between these defects is given by $\langle T^{00} \rangle$ as computed in section 3. From this, the force between the scatterers can be estimated. More importantly, our work sets the background for the study of adsorbed atoms in the nanocone surface, a subject of very high interest nowadays [48] since they are good candidates for gas storage. Adsorbed atoms become additional defects acting, again, as electron scatterers. Therefore, the fermionic Casimir effect with the inclusion of extra point defects gives the interaction between the adsorbed atoms and the apex or the boundary and among themselves [49]. This will be the subject of a forthcoming publication.

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